

ON FINITE p -GROUPS WHOSE CENTRAL AUTOMORPHISMS ARE ALL CLASS PRESERVING

MANOJ K. YADAV

ABSTRACT. We obtain certain results on a finite p -group whose central automorphisms are all class preserving. In particular, we prove that if G is a finite p -group whose central automorphisms are all class preserving, then $d(G)$ is even, where $d(G)$ denotes the number of elements in any minimal generating set for G . As an application of these results, we obtain some results regarding finite p -groups whose automorphisms are all class preserving.

1. INTRODUCTION

Let G be any group. By $Z(G)$, $\gamma_2(G)$ and $\Phi(G)$, we denote the center, the commutator subgroup and the Frattini subgroup of G respectively. For $x \in G$, $[x, G]$ denotes the set $\{[x, g] = x^{-1}g^{-1}xg \mid g \in G\}$ and x^G denotes the conjugacy class of x in G . Notice that $x^G = x[x, G]$ and therefore $|x^G| = |[x, G]|$ for all $x \in G$. For $x \in G$, $C_H(x)$ denotes the centralizer of x in H , where H is a subgroup of G . To say that some H is a subgroup (proper subgroup) of G we write $H \leq G$ ($H < G$). For any group H and an abelian group K , $\text{Hom}(H, K)$ denotes the group of all homomorphisms from H to K . For a finite p -group G , we denote by $\Omega_m(G)$ the subgroup $\langle x \in G \mid x^{p^m} = 1 \rangle$ and by Ω^m (which is not a standard notation) the subgroup $\langle x^{p^m} \mid x \in G \rangle$, where p is a prime integer and m is a positive integer. Let $d(G)$ denote the number of elements in any minimal generating set for a finite p -group G .

An automorphism ϕ of a group G is called *central* if $g^{-1}\phi(g) \in Z(G)$ for all $g \in G$. The set of all central automorphisms of G , denoted by $\text{Autcent}(G)$, is a normal subgroup of $\text{Aut}(G)$. Notice that $\text{Autcent}(G) = C_{\text{Aut}(G)}(\text{Inn}(G))$, where $\text{Inn}(G)$ denotes the group of all inner automorphisms of G . An automorphism α of G is called *class preserving* if $\alpha(x) \in x^G$ for all $x \in G$. The set of all class preserving automorphisms of G , denoted by $\text{Aut}_c(G)$, is a normal subgroup of $\text{Aut}(G)$. Notice that $\text{Inn}(G)$ is a normal subgroup of $\text{Aut}_c(G)$.

In 1999, A. Mann [17, Question 10] asked the following question: *Do all p -groups have automorphisms that are not class preserving? If the answer is no, which are the groups that have only class preserving automorphisms?* The first part of the question have a negative answer. The examples of finite p -groups G such that $\text{Aut}(G) = \text{Aut}_c(G)$ are already known in the literature. Such groups G , having nilpotency class 2 were constructed by H. Heineken [12] in 1980 and that having nilpotency class 3 were constructed by I. Malinowska [16] in 1992. So the second part of the question of Mann becomes relevant. Let us modify the question of Mann to make it more precise in the present scenario.

Question. Let $n \geq 4$ be a positive integer and p be a prime number. Does there exist a finite p -group of nilpotency class n such that $\text{Aut}(G) = \text{Aut}_c(G)$.

The second part of Mann's question, which clearly talks about the classification, can be stated as

Problem (A. Mann). Study (Classify) finite p -groups G such that $\text{Aut}(G) = \text{Aut}_c(G)$.

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In this paper we consider this problem for p -groups of nilpotency class 2 and make the stone rolling. Notice that, for a finite p -group of class 2, we have the following sequence of subgroups

$$\text{Aut}_c(G) \leq \text{Autcent}(G) \leq \text{Aut}(G).$$

The groups G such that $\text{Autcent}(G) = \text{Aut}(G)$, have been studied, every now and then, by many mathematicians (see [13] and [14] for recent developments and other references). So for studying groups G of nilpotency class 2 such that $\text{Aut}(G) = \text{Aut}_c(G)$, one needs to concentrate on the groups G satisfying

Hypothesis A. $\text{Aut}_c(G) = \text{Autcent}(G)$.

Suppose that $\text{Aut}_c(G) = \text{Autcent}(G)$, for some finite group G . Then $\text{Inn}(G) \leq \text{Autcent}(G)$. It then follows from the definition of $\text{Autcent}(G)$ that $\text{Inn}(G)$ is abelian. Hence the nilpotency class of G is at the most 2. Since a finite nilpotent group G can be written as a direct product of its Sylow p -subgroups, where p is a prime, to study $\text{Autcent}(G)$, it is sufficient to study the group of central automorphisms of finite p -group for the relevant prime integers p . Suppose that G is abelian and satisfies Hypothesis A, then $\text{Aut}_c(G) = 1$ and $\text{Autcent}(G) = \text{Aut}(G)$. Thus $\text{Aut}(G) = 1$. But this is possible only when $|G| \leq 2$. So from now onwards, we concentrate on finite p -groups of class 2, where p is a prime integer.

Let G be a finite p -group of class 2. Then $G/\text{Z}(G)$ is abelian. Consider the following cyclic decomposition of $G/\text{Z}(G)$.

$$G/\text{Z}(G) = \text{C}_{p^{m_1}} \times \cdots \times \text{C}_{p^{m_d}}$$

such that $m_1 \geq m_2 \geq \cdots \geq m_d \geq 1$, where $\text{C}_{p^{m_i}}$ denotes the cyclic group of order p^{m_i} for $1 \leq i \leq d$. The integers p^{m_1}, \dots, p^{m_d} are unique for $G/\text{Z}(G)$ and these are called the *invariants* of $G/\text{Z}(G)$. Now we state our first result in the following theorem, which we prove in Section 3 as Theorem 3.12.

Theorem A. *Let G be a finite p -group of class 2 and p^{m_1}, \dots, p^{m_d} be the invariants of $G/\text{Z}(G)$. Then G satisfies Hypothesis A if and only if $\gamma_2(G) = \text{Z}(G)$ and $|\text{Aut}_c(G)| = \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(G))|$.*

Our next result is the following theorem, which we prove in Section 3 as Theorem 3.15.

Theorem B. *Let G be a finite p -group of class 2 satisfying Hypothesis A. Then $d(G)$ is even.*

In the last section we concentrate on finite p -groups whose automorphisms are all class preserving and prove the following result.

Theorem C. *Let G be a non-abelian finite p -group such that $\text{Aut}(G) = \text{Aut}_c(G)$, where p is an odd prime. Then the following statements hold true.*

- (1.1a) $\gamma_2(G)$ cannot be cyclic.
- (1.1b) If $\text{Aut}(G)$ is elementary abelian, then G is a Camina special p -group.
- (1.1c) If $\text{Aut}(G)$ is abelian, then $d(G)$ is even.
- (1.1d) If $\text{Aut}(G)$ is abelian, then $|G| \geq p^8$ and $|\text{Aut}(G)| \geq p^{12}$.
- (1.1e) With $\text{Aut}(G)$ abelian, $|\text{Aut}(G)| = p^{12}$ if and only if $|G| = p^8$.
- (1.1f) If $\text{Aut}(G)$ is abelian of order p^{12} , then $\text{Aut}(G)$ is elementary abelian
- (1.1g) There exists a group G of order 3^8 such that $|\text{Aut}(G)| = |\text{Aut}_c(G)| = 3^{12}$.

In Section 2, we collect some basic results, which are useful for our work. Some further properties and some examples of finite groups satisfying Hypothesis A are obtained in Section 4.

We conclude this section with some definitions. A subset $\{y_1, \dots, y_d\}$ of a finite abelian group Y is said to be a *minimal basis* for Y if

$$Y = \langle y_1 \rangle \times \langle y_2 \rangle \times \cdots \times \langle y_d \rangle \quad \text{and} \quad |\langle y_1 \rangle| \geq |\langle y_2 \rangle| \geq \cdots \geq |\langle y_d \rangle| > 1.$$

A minimal generating set $\{x_1, \dots, x_d\}$ of a finite p -group G of nilpotency class 2 is said to be *distinguished* if the set $\{\bar{x}_1, \dots, \bar{x}_d\}$, $\bar{x}_i = x_i \text{Z}(G)$, forms a minimal basis for $G/\text{Z}(G)$.

2. SOME PREREQUISITES AND USEFUL LEMMAS

Let G be a finite group. Let $\alpha \in \text{Autcent}(G)$. Then the map f_α from G into $Z(G)$ defined by $f_\alpha(x) = x^{-1}\alpha(x)$ is a homomorphism which sends $\gamma_2(G)$ to 1. Thus f_α induces a homomorphism from $G/\gamma_2(G)$ into $Z(G)$. So we get a one-to-one map $\alpha \rightarrow f_\alpha$ from $\text{Autcent}(G)$ into $\text{Hom}(G/\gamma_2(G), Z(G))$. Conversely, if $f \in \text{Hom}(G/\gamma_2(G), Z(G))$, then α_f such that $\alpha_f(x) = xf(\bar{x})$ defines an endomorphism of G , where $\bar{x} = x\gamma_2(G)$. But this, in general, may not be an automorphism of G . More precisely, α_f fails to be an automorphism of G when G admits a non-trivial abelian direct factor.

A group G is called *purely non-abelian* if it does not have a non-trivial abelian direct factor.

The following theorem of Adney and Yen [1] shows that if G is a purely non-abelian finite group, then the mapping $\alpha \rightarrow f_\alpha$ from $\text{Autcent}(G)$ into $\text{Hom}(G/\gamma_2(G), Z(G))$, defined above, is also onto.

Theorem 2.1 ([1], Theorem 1). *Let G be a purely non-abelian finite group. Then the correspondence $\alpha \rightarrow f_\alpha$ defined above is a one-to-one mapping of $\text{Autcent}(G)$ onto $\text{Hom}(G/\gamma_2(G), Z(G))$.*

The following lemma follows from [1, page 141].

Lemma 2.2. *Let G be a finite p -group of class 2 such that $Z(G) = \gamma_2(G)$. Then $\text{Autcent}(G)$ is abelian.*

The following five lemmas, which we, sometimes, may use without any further reference, are well known.

Lemma 2.3. *Let C_n and C_m be two cyclic groups of order n and m respectively. Then $\text{Hom}(C_n, C_m) \cong C_d$, where d is the greatest common divisor of n and m , and C_d is the cyclic group of order d .*

Lemma 2.4. *Let A , B and C be finite abelian groups. Then*

- (i) $\text{Hom}(A \times B, C) \cong \text{Hom}(A, C) \times \text{Hom}(B, C)$;
- (ii) $\text{Hom}(A, B \times C) \cong \text{Hom}(A, B) \times \text{Hom}(A, C)$.

Lemma 2.5. *Let A , B and C be finite abelian groups such that A and B are isomorphic. Then $\text{Hom}(A, C) \cong \text{Hom}(B, C)$.*

Lemma 2.6. *Let A and C be finite abelian groups and B is a proper subgroup of C . Then $|\text{Hom}(A, B)| \leq |\text{Hom}(A, C)|$.*

Lemma 2.7. *Let C_{p^m} be a cyclic group of order p^m and B be any finite abelian group. Then $|\text{Hom}(C_{p^m}, B)| = |\text{Hom}(C_{p^m}, \Omega_m(B))|$.*

The following lemma seems well known. But we include a proof here, because we could not find a suitable reference for it.

Lemma 2.8. *Let G be a finite abelian p -group and M be a maximal subgroup of G . Then there exists a subgroup H of G and a positive integer i such that $G = H \times C_{p^{i+1}}$ and $M = H \times C_{p^i}$.*

Proof. We prove the lemma by induction on the finite order $|G| \geq p$ of G . Notice that $|G| = 1$ is impossible because G has a subgroup $M < G$. The lemma holds trivially when $|G| = p$. So we may assume that $|G| > p$, and that the lemma holds for all strictly smaller values of $|G|$.

Let $q = p^e$ be the exponent of G . Notice that $q > 1$ since $G \neq 1$. Any element $y \in G$ with order q lies in a minimal basis for G . So there is some subgroup I of G such that $G = \langle y \rangle \times I$. Furthermore $|I| < |G|$ since y has order $q > 1$.

Suppose that M contains the above element y of order q . Then $M = \langle y \rangle \times (M \cap I)$, where $M \cap I$ is a maximal subgroup of I . By induction there exist a subgroup $J < I$ and an element $x \in I$ such that $I = J \times \langle x \rangle$ and $M \cap I = J \times \langle x^p \rangle$. The lemma now holds with this x and the subgroup $H = J \times \langle y \rangle$ of G .

We have handled every case where M contains some element y of order q . So we may assume from now on that no such y lies in M . Let x_1, x_2, \dots, x_d be a minimal basis for G . Then the

order of x_1 is the exponent q of G . Since M contains no element of order q in G , it is contained in the subgroup

$$K = \langle x_1^p \rangle \times \langle x_2 \rangle \times \cdots \times \langle x_d \rangle,$$

which has index p in G . Since M is maximal in G , it must equal K . Then the lemma holds with $x = x_1$ and $H = \langle x_2 \rangle \times \cdots \times \langle x_d \rangle$. \square

The following interesting lemma is from [5, Lemma D].

Lemma 2.9. *Let A, B, C and D be finite abelian p -groups such that A is isomorphic to a proper subgroup of B and C is isomorphic to a proper subgroup of D . Then $|\text{Hom}(A, C)| < |\text{Hom}(B, D)|$.*

Proof. By Lemma 2.6, it is sufficient to prove the result when A is isomorphic to a maximal subgroup M of B and C is isomorphic to a maximal subgroup N of D . By Lemma 2.8, we have $M \cong E \times C_{p^i}$ and $B \cong E \times C_{p^{i+1}}$ for some group E and some non-negative integer i . Also $N \cong F \times C_{p^j}$ and $D \cong F \times C_{p^{j+1}}$ for some group F and some non-negative integer j . By Lemma 2.4

$$\text{Hom}(M, N) \cong \text{Hom}(E, F) \times \text{Hom}(E, C_{p^j}) \times \text{Hom}(F, C_{p^i}) \times \text{Hom}(C_{p^i}, C_{p^j})$$

and

$$\text{Hom}(B, D) \cong \text{Hom}(E, F) \times \text{Hom}(E, C_{p^{j+1}}) \times \text{Hom}(F, C_{p^{i+1}}) \times \text{Hom}(C_{p^{i+1}}, C_{p^{j+1}}).$$

Since $|\text{Hom}(E, C_{p^j})| \leq |\text{Hom}(E, C_{p^{j+1}})|$, $|\text{Hom}(F, C_{p^i})| \leq |\text{Hom}(F, C_{p^{i+1}})|$ and $|\text{Hom}(C_{p^i}, C_{p^j})| < |\text{Hom}(C_{p^{i+1}}, C_{p^{j+1}})|$, it follow that $|\text{Hom}(M, N)| < |\text{Hom}(B, D)|$. Now by Lemma 2.5 we get $\text{Hom}(M, N) \cong \text{Hom}(A, C)$. Hence $|\text{Hom}(A, C)| < |\text{Hom}(B, D)|$. \square

3. GROUPS G SATISFYING HYPOTHESIS A

In this section we derive some interesting properties of finite groups satisfying Hypothesis A and prove Theorems A and B. We start with the following easy lemma.

Lemma 3.1. *Let G be a finite p -group of class 2. Then the following holds:*

- (1) *The exponents of $\gamma_2(G)$ and $G/Z(G)$ are same.*
- (2) *For each $x \in G - Z(G)$, $[x, G]$ is a non-trivial normal subgroup of G contained in $\gamma_2(G)$.*
- (3) *For $x \in G - Z(G)$, the exponent of the subgroup $[x, G]$ is equal to the order of $\bar{x} = xZ(G) \in G/Z(G)$.*

Proof. Since (1) and (2) are well known, we only prove (3). Let the order of $\bar{x} = xZ(G)$ be p^c . Then $x^{p^c} \in Z(G)$. Let $[x, g] \in [x, G]$ be an arbitrary element. Now $[x, g]^{p^c} = [x^{p^c}, g] = 1$. Thus the exponent of $[x, G]$ is less than or equal to p^c . We claim that it can not be less than p^c . Suppose that the exponent of $[x, G]$ is p^b , where $b < c$. Then $[x^{p^b}, g] = [x, g]^{p^b} = 1$ for all $g \in G$. This proves that $x^{p^b} \in Z(G)$, which gives a contradiction to the fact that order of \bar{x} is p^c . Hence exponent of $[x, G]$ is equal to p^c , which completes the proof of the lemma. \square

Let G be a finite nilpotent group of class 2. Let $\phi \in \text{Aut}_c(G)$. Then the map $g \mapsto g^{-1}\phi(g)$ is a homomorphism of G into $\gamma_2(G)$. This homomorphism sends $Z(G)$ to 1. So it induces a homomorphism $f_\phi: G/Z(G) \rightarrow \gamma_2(G)$, sending $gZ(G)$ to $g^{-1}\phi(g)$, for any $g \in G$. It can be easily seen that the map $\phi \mapsto f_\phi$ is a monomorphism of the group $\text{Aut}_c(G)$ into $\text{Hom}(G/Z(G), \gamma_2(G))$.

Any $\phi \in \text{Aut}_c(G)$ sends any $g \in G$ to some $\phi(g) \in g^G$. Then $f_\phi(gZ(G)) = g^{-1}\phi(g)$ lies in $g^{-1}g^G = [g, G]$. Denote

$$\{ f \in \text{Hom}(G/Z(G), \gamma_2(G)) \mid f(gZ(G)) \in [g, G], \text{ for all } g \in G \}$$

by $\text{Hom}_c(G/Z(G), \gamma_2(G))$. Then $f_\phi \in \text{Hom}_c(G/Z(G), \gamma_2(G))$ for all $\phi \in \text{Aut}_c(G)$. On the other hand, if $f \in \text{Hom}_c(G/Z(G), \gamma_2(G))$, then the map sending any $g \in G$ to $gf(gZ(G))$ is an automorphism $\phi \in \text{Aut}_c(G)$ such that $f_\phi = f$. Thus we have

Proposition 3.2. *Let G be a finite nilpotent group of class 2. Then the above map $\phi \mapsto f_\phi$ is an isomorphism of the group $\text{Aut}_c(G)$ onto $\text{Hom}_c(G/Z(G), \gamma_2(G))$.*

We also need the following easy observation.

Lemma 3.3. *Let $H = C_{n_1} \times \cdots \times C_{n_r}$ and $K = C_{m_1} \times \cdots \times C_{m_s}$ be two finite abelian groups. Let $r \leq s$ and n_i divides m_i , where $1 \leq i \leq r$. Then H is isomorphic to a subgroup of K .*

We use above information and Lemma 2.9 to prove the following.

Proposition 3.4. *Let G be a finite p -group of class 2 which satisfies Hypothesis A. Then $\gamma_2(G) = Z(G)$.*

Proof. We first prove that G is purely non-abelian. Suppose the contrary, then $G = K \times A$, where A is a non-trivial abelian subgroup of G . Obviously $|\text{Aut}_c(G)| = |\text{Aut}_c(K)|$. But $|\text{Autcent}(G)| \geq |\text{Autcent}(K)| |\text{Autcent}(A)| > |\text{Autcent}(K)|$, since A is non-trivial. Hence

$$|\text{Aut}_c(G)| = |\text{Aut}_c(K)| \leq |\text{Autcent}(K)| < |\text{Autcent}(G)|.$$

This is a contradiction to Hypothesis A. This proves that G is purely non-abelian.

Now suppose that $\gamma_2(G) < Z(G)$. Then $|G/Z(G)| < |G/\gamma_2(G)|$. Notice that all the conditions of Lemma 3.3 hold with $H = G/Z(G)$ and $K = G/\gamma_2(G)$. Thus $G/Z(G)$ is isomorphic to a proper subgroup of $G/\gamma_2(G)$. It follows from Theorem 2.1 that $|\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))|$, since G is purely non-abelian. By Proposition 3.2, we have $|\text{Aut}_c(G)| \leq |\text{Hom}(G/Z(G), \gamma_2(G))|$. Since $G/Z(G)$ is isomorphic to a proper subgroup of $G/\gamma_2(G)$ and $\gamma_2(G)$ is a proper subgroup of $Z(G)$, it follows from Lemma 2.9 that

$$|\text{Hom}(G/Z(G), \gamma_2(G))| < |\text{Hom}(G/\gamma_2(G), Z(G))|.$$

Hence

$$|\text{Aut}_c(G)| \leq |\text{Hom}(G/Z(G), \gamma_2(G))| < |\text{Hom}(G/\gamma_2(G), Z(G))| = |\text{Autcent}(G)|.$$

This contradicts the fact that G satisfies Hypothesis A. Hence $\gamma_2(G) = Z(G)$. This completes the proof of the proposition. \square

Let G be a finite p -group of class 2 such that $Z(G) \leq \Phi(G)$. Then $\bar{G} = G/Z(G)$, being finite abelian, admits a minimal basis. Let $\{\bar{x}_1, \dots, \bar{x}_d\}$ be a minimal basis for \bar{G} , where $\bar{x}_i = x_i Z(G)$. Now $G/\Phi(G) \cong (G/Z(G))/(\Phi(G)/Z(G)) \cong \bar{G}/\Phi(\bar{G})$, since $Z(G) \leq \Phi(G)$. Thus the set $\{x_1\Phi(G), \dots, x_d\Phi(G)\}$ minimally generates $G/\Phi(G)$, which implies that the set $\{x_1, \dots, x_d\}$ minimally generates G . Now let $x \in G - \Phi(G)$. Therefore $xZ(G) \in G/Z(G) - \Phi(G)/Z(G)$. Thus we can find a minimal basis $\{\bar{x}_1, \dots, \bar{x}_d\}$ for \bar{G} such that $\bar{x} = xZ(G) = \bar{x}_i$ for some $1 \leq i \leq d$. As a consequence of this discussion, we get the following result.

Lemma 3.5. *Let G be a finite p -group of class 2 such that $Z(G) \leq \Phi(G)$. Then the following holds true:*

- (1) *Any minimal basis $\{\bar{x}_1, \dots, \bar{x}_d\}$ for \bar{G} provides a distinguished minimal generating set $\{x_1, \dots, x_d\}$ for G .*
- (2) *Any element $x \in G - \Phi(G)$ can be included in a distinguished minimal generating set for G .*

Since $Z(G) = \gamma_2(G) \leq \Phi(G)$ for a finite p -group of class 2 satisfying Hypothesis A, we readily get

Corollary 3.6. *Any finite p -group of class 2 satisfying Hypothesis A, admits a distinguished minimal generating set.*

Proposition 3.7. *Let G be a finite p -group of class 2 satisfying Hypothesis A. Let $\{x_1, \dots, x_d\}$ be a distinguished minimal generating set for G . Then $|\text{Aut}_c(G)| = \prod_{i=1}^d |x_i^G|$ and $[x_i, G] = \Omega_{m_i}(\gamma_2(G))$, where p^{m_i} is the order of \bar{x}_i . Moreover, $[x_1, G] = \gamma_2(G)$.*

Proof. Let $\{x_1, \dots, x_d\}$ be a distinguished minimal generating set for G such that order of \bar{x}_i is p^{m_i} for $1 \leq i \leq d$. Notice that $|\text{Aut}_c(G)| \leq \prod_{i=1}^d |x_i^G|$, as there are not more than $|x_i^G|$ choices for the image of x_i under any class preserving automorphism of G . Since the exponent of the

subgroup $[x_i, G]$ is equal to the order of $\bar{x}_i = x_i Z(G) \in G/Z(G)$ for any $1 \leq i \leq d$ (Lemma 3.1), it follows that $|\text{Hom}(\langle \bar{x}_i \rangle, [x_i, G])| = |[x_i, G]|$. By Proposition 3.4 we have $Z(G) = \gamma_2(G)$. Thus

$$\begin{aligned} |\text{Aut}_c(G)| &= |\text{Autcent}(G)| = |\text{Hom}(G/\gamma_2(G), Z(G))| = |\text{Hom}(G/Z(G), \gamma_2(G))| \\ (3.8) \quad &= \prod_{i=1}^d |\text{Hom}(\langle \bar{x}_i \rangle, \gamma_2(G))| \geq \prod_{i=1}^d |\text{Hom}(\langle \bar{x}_i \rangle, [x_i, G])| = \prod_{i=1}^d |[x_i, G]| \\ &= \prod_{i=1}^d |x_i^G|. \end{aligned}$$

Hence $|\text{Aut}_c(G)| = \prod_{i=1}^d |x_i^G|$.

It now follows from (3.8) that $\text{Hom}(\langle \bar{x}_i \rangle, \gamma_2(G)) = \text{Hom}(\langle \bar{x}_i \rangle, [x_i, G])$ for each $1 \leq i \leq d$. Notice that $\text{Hom}(\langle \bar{x}_i \rangle, \gamma_2(G)) = \text{Hom}(\langle \bar{x}_i \rangle, \Omega_{m_i}(\gamma_2(G))) \cong \Omega_{m_i}(\gamma_2(G))$ (Lemma 2.7). Also, as mentioned above, $\text{Hom}(\langle \bar{x}_i \rangle, [x_i, G]) \cong [x_i, G]$. Hence $[x_i, G] = \Omega_{m_i}(\gamma_2(G))$. Since the exponent of $[x_i, G]$ is equal to the order of \bar{x}_i , it follows that $[x_i, G] \leq \Omega_{m_i}(\gamma_2(G))$ for each $1 \leq i \leq d$. Hence $[x_i, G] = \Omega_{m_i}(\gamma_2(G))$ for each $1 \leq i \leq d$. Since the order of \bar{x}_1 (which is equal to the exponent of $G/Z(G)$) is equal to the exponent of $\gamma_2(G)$, $[x_1, G] = \gamma_2(G)$. This completes the proof of the proposition. \square

In the following corollary we show that the order of $\text{Aut}_c(G)$, obtained in Proposition 3.7, is independent of the choice of distinguished minimal generating set for G .

Corollary 3.9. *Let G be a finite p -group of class 2 satisfying Hypothesis A. Let the invariants of $G/Z(G)$ be p^{m_1}, \dots, p^{m_d} . Then $|\text{Aut}_c(G)| = \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(G))|$.*

Proof. Let p^{m_1}, \dots, p^{m_d} be the invariants of $G/Z(G)$. Notice that any distinguished minimal generating set for G can be written (after re-ordering if necessary) as $\{x_1, \dots, x_d\}$ such that order of \bar{x}_i is p^{m_i} for $1 \leq i \leq d$. Hence, by Proposition 3.7, it follows that $|\text{Aut}_c(G)| = \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(G))|$. This completes the proof. \square

Some other consequences of Proposition 3.7 are

Corollary 3.10. *Let G be a finite p -group of class 2 satisfying Hypothesis A. Let $x \in G - \Phi(G)$ such that order of $xZ(G)$ is p^m . Then $[x, G] = \Omega_m(\gamma_2(G))$.*

Proof. By Lemma 3.5, we can find a distinguished minimal generating set $\{x_1, \dots, x_d\}$ for G such that $x = x_i$ for some $1 \leq i \leq d$. Now the assertion follows from Proposition 3.7. \square

Corollary 3.11. *Let G be a finite p -group of class 2 satisfying Hypothesis A. Let $x \in G - \Phi(G)$ such that order of $xZ(G)$ is p^m . Then $|C_G(x)| = |G/Z(G)| |\Omega^m(\gamma_2(G))|$.*

We now prove Theorem A.

Theorem 3.12 (Theorem A). *Let G be a finite p -group of nilpotency class 2. Then G satisfies Hypothesis A if and only if $Z(G) = \gamma_2(G)$ and $|\text{Aut}_c(G)| = \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(G))|$, where p^{m_1}, \dots, p^{m_d} are the invariants of $G/Z(G)$.*

Proof. Let G satisfy Hypothesis A. Then by Proposition 3.4, $Z(G) = \gamma_2(G)$ and by Corollary 3.9, $|\text{Aut}_c(G)| = \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(G))|$.

On the other hand, suppose that $Z(G) = \gamma_2(G)$ and $|\text{Aut}_c(G)| = \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(G))|$. Let $\{x_1, \dots, x_d\}$ be a distinguished minimal generating set for G such that order of $\bar{x}_i = x_i Z(G)$ is p^{m_i} for $1 \leq i \leq d$. Since $Z(G) = \gamma_2(G)$, G is purely non-abelian and therefore

$$\begin{aligned} |\text{Autcent}(G)| &= \prod_{i=1}^d |\text{Hom}(\langle \bar{x}_i \rangle, Z(G))| = \prod_{i=1}^d |\text{Hom}(\langle \bar{x}_i \rangle, \Omega_{m_i}(\gamma_2(G)))| \\ &= \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(G))| = |\text{Aut}_c(G)|. \end{aligned}$$

Hence G satisfies hypothesis A. This completes the proof of the theorem. \square

Let $Z(G) = Z_1 \times Z_2 \times \dots \times Z_r$ be a cyclic decomposition of $Z(G)$ such that $|Z_1| \geq |Z_2| \geq \dots \geq |Z_r| > 1$ and, for each i such that $1 \leq i \leq r$, define $Z_i^* = Z_1 \times \dots \times Z_{i-1} \times Z_{i+1} \times \dots \times Z_r$.

Proposition 3.13. *Let G be a finite p -group of class 2 satisfying Hypothesis A. Let $x \in G - \Phi(G)$. Then $[x, G] \cap Z_i \neq 1$ for each i such that $1 \leq i \leq r$. Moreover, if $|Z_i|$ is equal to the exponent of $Z(G)$, then $[x, G] \cap Z_i$ is equal to the exponent of $[x, G]$ and G/Z_i^* satisfies Hypothesis A.*

Proof. Since G satisfies Hypothesis A, $Z(G) = \gamma_2(G)$. We'll make use of this fact without telling so. Let $x \in G - \Phi(G)$ be such that order of $xZ(G)$ is p^m , where $m \geq 1$ is an integer. Then it follows from Corollary 3.10 that $[x, G] = \Omega_m(\gamma_2(G)) = \Omega_m(Z(G))$. This proves that $[x, G] \cap Z_i \neq 1$ for each $1 \leq i \leq r$.

Let $|Z_i| = p^e$, the exponent of $Z(G)$. Since the exponent of $[x, G]$ is equal to $|\langle \bar{x} \rangle|$, where $\bar{x} = xZ(G)$ (Lemma 3.1), the exponent of $[x, G]$ is p^m . Thus $e \geq m$. Since $[x, G] = \Omega_m(Z(G))$, it follows that $[x, G] \cap Z_i = \Omega_m(Z_i)$. Hence $|[x, G] \cap Z_i| = p^m$.

Again let $|Z_i| = p^e$, the exponent of $Z(G)$. Assume for a moment that $Z(G/Z_i^*) = Z(G)/Z_i^*$. Then $Z(G/Z_i^*) \cong Z_i$ is cyclic and is equal to $\gamma_2(G/Z_i^*) = \gamma_2(G)/Z_i^*$, since $Z_i^* \leq \gamma_2(G)$. This implies that $\text{Autcent}(G/Z_i^*) = \text{Inn}(G/Z_i^*) = \text{Aut}_c(G/Z_i^*)$. Hence Hypothesis A holds true for G/Z_i^* . Therefore, to complete the proof, it is sufficient to prove what we have assumed, i.e., $Z(G/Z_i^*) = Z(G)/Z_i^*$.

Let $xZ_i^* \in G/Z_i^* - Z(G)/Z_i^*$. Then $x \in G - Z(G)$. If $x \in G - \Phi(G)$, then $[x, G] \not\subseteq Z_i^*$ and therefore $xZ_i^* \notin Z(G/Z_i^*)$. So let $x \in \Phi(G) - Z(G)$. Then there exists an element $y \in G - \Phi(G)$ such that $x = y^j z$ for some positive integer j and some element $z \in Z(G)$. Now

$$[x, G] = [y^j z, G] = [y, G]^j, \quad 1 \leq j < \text{exponent of } [y, G].$$

Since $y \in G - \Phi(G)$, $[y, G] \cap Z_i$ is equal to the exponent of $[y, G]$. So it follows that $[y, G]^j \not\subseteq Z_i^*$ for any non-zero j which is strictly less than the exponent of $[y, G]$. Thus $[x, G] \not\subseteq Z_i^*$. This implies that $[xZ_i^*, G/Z_i^*] \neq 1$. Hence $xZ_i^* \notin Z(G/Z_i^*)$. This proves that $Z(G/Z_i^*) = Z(G)/Z_i^*$, completing the proof of the proposition. \square

For the proof of our next result we need the following important result.

Theorem 3.14 ([3], Theorem 2.1). *Let G be a finite p -group of nilpotency class 2 with cyclic center. Then G is a central product either of two generator subgroups with cyclic center or two generator subgroups with cyclic center and a cyclic subgroup.*

Now we are ready to prove Theorem B.

Theorem 3.15 (Theorem B). *Let G be a finite p -group of class 2 satisfying Hypothesis A. Then $d(G)$ is even.*

Proof. Let $|Z_i|$ be equal to the exponent of $Z(G)$ and G^* denote the factor group G/Z_i^* . Then it follows from Proposition 3.13 that $Z(G^*) = \gamma_2(G^*)$ is cyclic of order $|Z_i|$. Since G^* is purely non-abelian, it follows from Theorem 3.14 that G is a central product of two generator subgroups with cyclic center. Hence $G^*/Z(G^*) \cong G/Z(G)$ is a direct product of even number of cyclic subgroups of $G^*/Z(G^*)$. Thus $d(G/Z(G))$ is even. Since $Z(G) = \gamma_2(G)$, it follows that $d(G)$ is even. \square

4. SOME FURTHER PROPERTIES AND EXAMPLES

In this section we discuss some more properties and give some examples of finite groups which satisfy Hypothesis A. We start with the following concept of isoclinism of groups, introduced by P. Hall [10].

Let X be a finite group and $\bar{X} = X/Z(X)$. Then commutation in X gives a well defined map $a_X : \bar{X} \times \bar{X} \mapsto \gamma_2(X)$ such that $a_X(xZ(X), yZ(X)) = [x, y]$ for $(x, y) \in X \times X$. Two finite groups G and H are called *isoclinic* if there exists an isomorphism ϕ of the factor group $\bar{G} = G/Z(G)$ onto $\bar{H} = H/Z(H)$, and an isomorphism θ of the subgroup $\gamma_2(G)$ onto $\gamma_2(H)$ such that the following diagram is commutative

$$\begin{array}{ccc} \bar{G} \times \bar{G} & \xrightarrow{a_G} & \gamma_2(G) \\ \phi \times \phi \downarrow & & \downarrow \theta \\ \bar{H} \times \bar{H} & \xrightarrow{a_H} & \gamma_2(H). \end{array}$$

The resulting pair (ϕ, θ) is called an *isoclinism* of G onto H . Notice that isoclinism is an equivalence relation among finite groups.

Let G be a finite p -group. Then it follows from [10] that there exists a finite p -group H in the isoclinism family of G such that $Z(H) \leq \gamma_2(H)$. Such a group H is called a *stem group* in the isoclinism family of G .

The following theorem shows that the group of class preserving automorphisms is independent of the choice of a group in a given isoclinism family of groups.

Theorem 4.1 ([21], Theorem 4.1). *Let G and H be two finite non-abelian isoclinic groups. Then $\text{Aut}_c(G) \cong \text{Aut}_c(H)$.*

Proposition 4.2. *Let G and H be two non-abelian finite p -groups which are isoclinic. Let G satisfy Hypothesis A. Then H satisfies Hypothesis A if and only if $|H| = |G|$.*

Proof. Suppose that H satisfies Hypothesis A. Then $\gamma_2(H) = Z(H)$. Since G and H are isoclinic and G satisfies Hypothesis A, it follows that $|Z(G)| = |\gamma_2(G)| = |\gamma_2(H)| = |Z(H)|$ and $|G/Z(G)| = |H/Z(H)|$. Hence $|H| = |H/Z(H)||Z(H)| = |G/Z(G)||Z(G)| = |G|$.

Conversely, suppose that $|H| = |G|$. It is easy to show that $\gamma_2(H) = Z(H)$. Let p^{m_1}, \dots, p^{m_d} be the invariants of $G/Z(G) \cong H/Z(H)$. Since $\gamma_2(H) \cong \gamma_2(G)$, we have $\Omega_{m_i}(\gamma_2(H)) \cong \Omega_{m_i}(\gamma_2(G))$. Since G and H are isoclinic, it follows from Theorem 4.1 that $\text{Aut}_c(G) \cong \text{Aut}_c(H)$. Hence $|\text{Aut}_c(H)| = \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(G))| = \prod_{i=1}^d |\Omega_{m_i}(\gamma_2(H))|$. That H satisfies Hypothesis A, now follows from Theorem 3.12. \square

Let G be a finite group and $1 \neq N$ be a normal subgroup of G . (G, N) is called a *Camina pair* if $xN \subseteq x^G$ for all $x \in G - N$. A group G is called a *Camina group* if $(G, \gamma_2(G))$ is a Camina pair. So if G is a Camina group, then $x\gamma_2(G) \subseteq x^G$ all $x \in G - \gamma_2(G)$. This is equivalent to saying that $\gamma_2(G) \subseteq [x, G]$ all $x \in G - \gamma_2(G)$. Since $[x, G] \subseteq \gamma_2(G)$, it follows that G is a Camina group if and only if $\gamma_2(G) = [x, G]$ all $x \in G - \gamma_2(G)$.

Proposition 4.3. *Let G be a finite special p -group. Then Hypothesis A holds true for G if and only if G is a Camina group.*

Proof. First suppose that Hypothesis A holds true for G . Since $\gamma_2(G) = \Phi(G)$, we only need to show that $[x, G] = \gamma_2(G)$ for all $x \in G - \Phi(G)$. Let $x \in G - \Phi(G)$. By Corollary 3.10, we have $[x, G] = \Omega_m(\gamma_2(G))$, where order of $xZ(G)$ in $G/Z(G)$ is p^m . Since G is special, $m = 1$ and the exponent of $\gamma_2(G)$ is p . Thus $[x, G] = \gamma_2(G)$.

Conversely suppose that G is a Camina group. Then it follows from [20, Theorem 5.4] that $|\text{Aut}_c(G)| = |\gamma_2(G)|^d$, where $|G/\Phi(G)| = p^d$. Since G is special, it follows that $\gamma_2(G) = Z(G)$ and $p^{m_1} = p, p^{m_2} = p, \dots, p^{m_d} = p$ are the invariants of $G/Z(G)$. Thus $\gamma_2(G) = \Omega_{m_i}(\gamma_2(G))$. Now we can use Theorem 3.12 to deduce that G satisfies Hypothesis A and the proof is complete. \square

Since every finite Camina p -group of nilpotency class 2 is special [15], we readily get

Corollary 4.4. *Let G be a finite Camina p -group of nilpotency class 2. Then G satisfies Hypothesis A.*

With this much information we get

Example 1. All finite Camina p -groups of class 2 satisfy Hypothesis A. In particular, all finite extra-special p -groups satisfy Hypothesis A. Examples of non extra-special Camina p -groups of class 2 can be found in [6] and [15].

Now we construct an example of a finite p -group satisfying Hypothesis A, which is not a Camina group.

Example 2. Let R be the factor ring S/p^2S , where S is the ring of p -adic integers in the unramified extension of degree 2 over the p -adic completion \mathbb{Q}_p of the rational numbers \mathbb{Q} . Form the group G of all 3×3 matrices

$$M(x, y, z) = \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix}$$

for $x, y, z \in R$. The additive group of R is the direct sum of two copies of a cyclic group of order p^2 . The factor ring R/pR modulo the ideal pR is a finite field of order p^2 . Notice that commutation in G satisfies

$$[M(x, y, z), M(x', y', z')] = M(0, 0, yx' - xy')$$

for any $x, y, z, x', y', z' \in R$. So both the center $Z(G)$ and the derived group $\gamma_2(G)$ consist of all matrices of the form $M(0, 0, z)$ for $z \in R$. Since $M(0, 0, z)M(0, 0, z') = M(0, 0, z + z')$ for all $z, z' \in R$, $Z(G)$ is noncyclic and equal to $\gamma_2(G)$. Thus the nilpotency class of G is 2.

From the above formula for commutators it follows that

$$[M(x, y, z), G] = M(0, 0, Rx + Ry) := \{M(0, 0, z) \mid z \in Rx + Ry\}$$

for any $x, y, z \in R$. Note that there are only three choices for the ideal $Rx + Ry$ in R , namely, R , pR and $p^2R = 0$. Furthermore, all three possibilities happen for suitable x and y . Now

$$[M(1, 0, 0), G] = [M(p - 1, 0, 0), G] = M(0, 0, R) = Z(G)$$

and

$$[M(1, 0, 0)M(p - 1, 0, 0), G] = [M(p, 0, 0), G] = M(0, 0, pR) = Z(G)^p.$$

So

$$[M(1, 0, 0), G][M(p - 1, 0, 0), G] = Z(G) > Z(G)^p = [M(1, 0, 0)M(p - 1, 0, 0), G].$$

This shows that G has a non-central element $x = M(1, 0, 0)M(p - 1, 0, 0)$ such that $[x, G] < \gamma_2(G) = Z(G)$. Hence G is not a Camina group.

Since $Z(G) = \gamma_2(G)$, G is purely non-abelian. Then from Lemma 2.1, it follows that for any element $\alpha \in \text{Autcent}(G)$ there exists a corresponding element $f_\alpha \in \text{Hom}(G/\gamma_2(G), Z(G))$ such that $f_\alpha(\bar{x}) = x^{-1}\alpha(x)$ for each $\bar{x} \in G/\gamma_2(G)$. Let x, y, z be any three elements of R , and $i = 0, 1, 2$ be such that $Rx + Ry = p^iR$. Then the element $M(x, y, z)Z(G)$ of $G/Z(G)$ lies in $(G/Z(G))^{p^i}$. So its image $f(M(x, y, z)Z(G))$ lies in

$$Z(G)^{p^i} = M(0, 0, p^iR) = M(0, 0, Rx + Ry) = [M(x, y, z), G].$$

Thus $f(g) \in [g, G]$ for all $g \in G$, and $\alpha \in \text{Aut}_c(G)$. Since $\text{Aut}_c(G) \leq \text{Autcent}(G)$, this proves that $\text{Aut}_c(G) = \text{Autcent}(G)$.

5. GROUPS G WITH $\text{Aut}(G) = \text{Aut}_c(G)$

In this section we prove Theorem C. Throughout the section, p always denotes an odd prime. We state some important known results in the following theorem.

Theorem 5.1. *The following statements hold true.*

- (1) *Let G be a non-abelian p -group of order p^5 or less. Then $\text{Aut}(G)$ is non-abelian.* [7, Theorem 5.2]
- (2) *There is no group G of order p^6 whose automorphism group is an abelian p -group.* [18, Proposition 1.4]
- (3) *Let G be a non-abelian finite p -group such that $\text{Aut}(G)$ is abelian. Then $d(G) \geq 4$.* [19]
- (4) *Let G be a non-cyclic finite p -group, p odd, for which $\text{Aut}(G)$ is abelian. Then p^{12} divides $|\text{Aut}(G)|$.* [11, Main Theorem]
- (5) *Let G be a non-cyclic group of order p^7 . If $\text{Aut}(G)$ is abelian, then it must be of order p^{12} .* [2, Theorem 1]
- (6) *Let G be a finite non-cyclic p -group such that $\text{Aut}(G)$ is abelian. Then $\gamma_2(G) \cong C_{p^m} \times C_{p^m}$ or $C_{p^m} \times C_{p^{m_1}} \times \cdots \times C_{p^{m_k}}$, where p^m is the exponent of $\gamma_2(G)$ and $m \geq m_i$ for $1 \leq i \leq k$.* [2, Lemma 4(12)]
- (7) *Let G be a finite Camina p -group of class 2 such that $d(G) = n$ and $d(\gamma_2(G)) = m$ for some positive integers n and m . Then n is even and $n \geq 2m$. (Follows from [15, Theorems 3.1, 3.2])*

Now we start the proof of Theorem C.

Lemma 5.2. *Let G be a non-abelian finite p -group such that $\text{Aut}(G) = \text{Aut}_c(G)$, where p is an odd prime. Then $\gamma_2(G)$ cannot be cyclic.*

Proof. Suppose that $\gamma_2(G)$ is cyclic. It now follows from [4, Theorem 3] that $\text{Aut}(G) = \text{Aut}_c(G) = \text{Inn}(G)$, which is not possible by a celebrated theorem of W. Gaschütz [9]. \square

Lemma 5.3. *Let G be a non-abelian finite p -group such that $\text{Aut}(G) = \text{Aut}_c(G)$. Then $\text{Aut}(G)$ is elementary abelian if and only if G is a Camina special p -group.*

Proof. Suppose that $\text{Aut}(G) = \text{Aut}_c(G)$ is elementary abelian. Then $G/\text{Z}(G) \cong \text{Inn}(G)$ is elementary abelian. Since $\text{Z}(G) = \gamma_2(G) \leq \Phi(G)$, it follows that G is a special p -group. Hence, by Proposition 4.3, G is a Camina special p -group.

Conversely, suppose that G is a Camina p -group of class 2. Then $\text{Inn}(G) \cong G/\text{Z}(G)$ is elementary abelian. Hence it follows that $\text{Aut}(G) = \text{Aut}_c(G)$ is elementary abelian. \square

Remark 5.4. If G is a special p -group and $\text{Aut}(G) = \text{Autcent}(G)$, then it is not difficult to prove that $\text{Aut}(G)$ is elementary abelian. But the converse is not true. The examples of non-special finite p -groups G such that $\text{Aut}(G) = \text{Autcent}(G)$ is elementary abelian can be found in [14].

Proposition 5.5. *Let G be a finite p -group of nilpotency class 2 such that $\text{Aut}(G) = \text{Aut}_c(G)$, where p is an odd prime. Then $|G| \geq p^8$.*

Proof. Since the nilpotency class of G is 2, $\text{Aut}(G) = \text{Aut}_c(G)$ is abelian. It now follows from Theorem 5.1(1) and Theorem 5.1(2) that $|G| \geq p^7$. Let $|G| = p^7$. Then $|\text{Aut}(G)| = p^{12}$ (by Theorem 5.1(5)). Now using the fact that $\text{Z}(G) = \gamma_2(G)$ and $|\text{Aut}(G)| = |\text{Autcent}(G)| = |\text{Hom}(G/\text{Z}(G), \text{Z}(G))| = p^{12}$, it follows from Lemma 5.2 and Theorems 3.15, 5.1(3) and 5.1(4) (by looking various possibilities for the order and structure of $\text{Z}(G)$) that G is a special p -group with $|G/\text{Z}(G)| = p^4$ and $|\text{Z}(G)| = p^3$. It then follows from Proposition 4.3 that G is a Camina special p -group, which is not possible by Theorem 5.1(7). This completes the proof. \square

The following lemma seems basic.

Lemma 5.6. *Let G be a finite p -group of nilpotency class 2 such that $\gamma_2(G) \cong C_{p^m} \times C_{p^m}$ for some positive integer m . Then $G/\text{Z}(G) \cong C_{p^m} \times C_{p^m} \times C_{p^m} \times H$ for some abelian group H .*

Proof. Since the exponent of $\gamma_2(G)$ is p^m , we can write $G/\text{Z}(G) = \langle \bar{x}_1 \rangle \times \langle \bar{x}_2 \rangle \times K/\text{Z}(G)$ for some subgroup K of G containing $\text{Z}(G)$ such that $|\langle [x_1, x_2] \rangle| = |\langle \bar{x}_1 \rangle| = |\langle \bar{x}_2 \rangle| = p^m$. We only need to prove that the exponent of $K/\text{Z}(G)$ is p^m . Since $d(\gamma_2(G)) = 2$, we can find an element $w \in \gamma_2(G)$ such that $\gamma_2(G) = \langle [x_1, x_2], w \rangle$, where

$$w = [x_1, x_2]^a [x_1, k]^{a_1} [x_2, k']^{a_2} \prod_{k_i, k_j \in K} [k_i, k_j]^{b_{ij}}$$

for some $k, k', k_i, k_j \in K$. Let $u = [x_1, x_2]^{a_1} [k_1, x_2]^{a_2} \prod_{k_i, k_j \in K} [k_i, k_j]^{b_{ij}}$. Then notice that $[x_1, x_2]$ and u also generate $\gamma_2(G)$. If the exponent of $K/\text{Z}(G)$ is less than p^m , then order of the element u is also less than p^m . Thus $|\gamma_2(G)| < p^{2m}$, which is a contradiction to the given fact that $\gamma_2(G) \cong C_{p^m} \times C_{p^m}$. Hence the exponent of $K/\text{Z}(G)$ is p^m and the proof of the lemma is complete. \square

Proposition 5.7. *Let G be a finite p -group such that $\text{Aut}(G) = \text{Aut}_c(G)$ is abelian. Then $|\text{Aut}(G)| = p^{12}$ if and only if $|G| = p^8$.*

Proof. Let $|\text{Aut}(G)| = p^{12}$. By Proposition 5.5, we can assume that $|G| \geq p^8$. Also, by Theorem 3.15, $G/\text{Z}(G)$ is minimally generated by even number of elements. Notice that

$$|\text{Hom}(G/\text{Z}(G), \text{Z}(G))| = |\text{Autcent}(G)| = |\text{Aut}(G)| = p^{12}.$$

Since $\text{Z}(G) = \gamma_2(G)$ can not be cyclic, a copy of $C_p \times C_p$ is sitting inside $\text{Z}(G)$. Suppose that $d(\text{Z}(G)) \geq 3$. Since $d(G/\text{Z}(G)) \geq 4$ (by Theorem 5.1(3)), we have $|\text{Aut}(G)| \geq p^{12}$. Notice that the equality holds only when $d(\text{Z}(G)) = 3$, $d(G/\text{Z}(G)) = 4$ and the exponent of both $\text{Z}(G)$ and $G/\text{Z}(G)$ is p . This implies that $|G| = p^7$, which we are not considering. Thus $d(\text{Z}(G)) = 2$. Let

the exponent of $Z(G)$ is p^m for some positive integer m . Then it follows from Theorem 5.1(6) that $Z(G) \cong C_{p^m} \times C_{p^m}$. We claim that $m = 1$. Suppose that $m \geq 2$. If $d(G/Z(G)) \geq 6$, then notice that $|\text{Aut}(G)| > p^{12}$. Thus by Theorem 3.15, $d(G/Z(G)) = 4$ and by Lemma 5.6, $G/Z(G) \cong C_{p^m} \times C_{p^m} \times C_{p^r}$ for some $1 \leq r \leq m$. Now it is easy to show that $|\text{Aut}(G)| > p^{12}$. This contradiction proves our claim, i.e., $m = 1$. Now the only choice for $d(G/Z(G))$ to give $|\text{Aut}(G)| = p^{12}$ is 6. Thus $G/Z(G)$ and $Z(G)$ are elementary abelian of order p^6 and p^2 respectively. Hence $|G| = p^8$.

Conversely, suppose that $|G| = p^8$ and $\text{Aut}(G) = \text{Aut}_c(G)$ is abelian. Then $|Z(G)| = |\gamma_2(G)| \leq p^4$. If $|Z(G)| = p^4$, then $|G/Z(G)| = p^4$ and therefore $G/Z(G)$ is elementary abelian by Theorem 5.1(3). Hence $\Phi(G) = Z(G) = \gamma_2(G)$ is elementary abelian. This shows that G is special and therefore Camina special. But this is not possible by Theorem 5.1(7). If $|Z(G)| = p^3$, then by Theorem 3.15 and Theorem 5.1(3), $G/Z(G)$ can be written as

$$G/Z(G) \cong C_{p^2} \times C_p \times C_p \times C_p$$

and by Lemma 5.2, $Z(G)$ can be written as

$$Z(G) \cong C_{p^2} \times C_p.$$

Thus $|\text{Aut}(G)| = |\text{Autcent}(G)| = |\text{Hom}(G/Z(G), Z(G))| = p^9$, which is not possible by Theorem 5.1(5). Hence $|Z(G)| = p^2$. Now $Z(G)$, being non-cyclic by Lemma 5.2, must be isomorphic to $C_p \times C_p$. Since the nilpotency class of G is 2 and $Z(G) = \gamma_2(G)$, $G/Z(G)$ must be elementary abelian of order p^6 . Hence $|\text{Aut}(G)| = |\text{Autcent}(G)| = |\text{Hom}(G/Z(G), Z(G))| = p^{12}$. \square

It only remains to establish the final thread of Theorem C. For an odd prime p , consider the group

$$(5.8) \quad G = \langle x_1, x_2, x_3, x_4, x_5, x_6 \mid x_1^{p^2} = x_2^{p^2} = x_3^p = x_4^p = x_5^p = x_6^p = 1, \\ [x_1, x_2] = x_1^p, [x_1, x_3] = x_2^p, [x_2, x_3] = x_1^p, [x_1, x_4] = x_2^p, [x_2, x_4] = x_2^p, \\ [x_3, x_4] = x_2^p, [x_1, x_5] = x_2^p, [x_2, x_5] = x_1^p, [x_3, x_5] = x_2^p, [x_4, x_5] = x_1^p, \\ [x_1, x_6] = x_2^p, [x_2, x_6] = x_2^p, [x_3, x_6] = x_1^p, [x_4, x_6] = x_1^p, [x_5, x_6] = x_2^p \rangle$$

The following lemma completes the proof of Theorem C.

Lemma 5.9. *The group G , defined in (5.8), is a Camina special p -group of order p^8 with $|Z(G)| = p^2$ for all odd primes p . For $p = 3$, $\text{Aut}(G) = \text{Aut}_c(G)$ is elementary abelian of order p^{12} .*

Proof. It is fairly easy to show that G is a special p -group of order p^8 with $|Z(G)| = p^2$. Then by Theorem 2.1, it follows that $|\text{Autcent}(G)| = p^{12}$. For $p = 3$, using GAP [8] one can easily establish (i) G has 737 conjugacy classes and therefore it is a Camina group; (ii) $\text{Aut}(G)$ is elementary abelian of order p^{12} . Thus, by Proposition 4.3, G satisfies Hypothesis A. Hence $|\text{Aut}(G)| = |\text{Aut}_c(G)| = p^{12}$. \square

We conclude this section with some questions. A finite abelian group Y of exponent e is said to be *homocyclic* if the set of its invariants is $\{e\}$.

Question 1. Let G be a finite p -group of class 2 such that $\text{Aut}(G) = \text{Aut}_c(G)$. Is it true that $d(G) \geq 2d(\gamma_2(G))$? If the answer is negative, what is the relationship between $d(G)$ and $d(\gamma_2(G))$?

Question 2. Let G be a finite p -group of class 2 such that $\text{Aut}(G) = \text{Aut}_c(G)$ and $G/Z(G)$ is homocyclic. Is $\gamma_2(G)$ homocyclic? If not, how big homocyclic group of the highest exponent, $\gamma_2(G)$ contains?

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SCHOOL OF MATHEMATICS, HARISH-CHANDRA RESEARCH INSTITUTE, CHHATNAG ROAD, JHUNSI, ALLAHABAD - 211 019, INDIA

E-mail address: myadav@mri.ernet.in